Ordinary Differential Equations.

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Introduction. This is a report from the working group charged with making recommendations for the undergraduate curriculum in differential equations. As discussed below, the basic sophomore level Differential Equations course has changed dramatically (at least at some institutions) over the past twenty years, mainly due to the much wider availiblility of computer resources. We discuss a number of these changes below and then provide three different syllabi, each aimed at a slightly different audience. Finally, we provide some references including sample texts.

How Differential Equations Courses Have Evolved. As part of our committee work, we surveyed a half dozen different schools regarding their sophomore-level Ordinary Differential Equations (ODE) course. These schools included two-year colleges, liberal arts colleges, and universities. Two things became clear from this survey. The first is that the ODE course has undergone a remarkable transformation over the past twenty years. The second is that there is now no set curriculum for this course.

Regarding the first point, the ODE course previously focused almost exclusively on specific analytic methods for solving differential equations. Often, examples of differential equations were simplified so that the corresponding equation could then be solved explicitly (as, for example, the nonlinear pendulum equation was often linearized to accomplish this). Usually, the types of equations covered in the course were first and second (and maybe higher) order differential equations but rarely included systems of differential equations.

The major change in the ODE courses over the past two decades has been primarily motivated by the availability of the computer. Currently, most courses use some of the many different software packages that are available to approximate and visualize solutions of ODEs. Differential equations that are completely unsolvable using analytic techniques can now be approached from a geometric point of view using tools from dynamical systems. These include important differential equations that arise in many different areas of science and engineering, topics that were untreatable in this course a few years back. This has led to many new topics being introduced into this course. As a consequence, some specialized analytical techniques for solving ODEs have necessarily been dropped from the course.

The ODE Audience. At many institutions, the ODE course is often the final mathematics course required by students in engineering and certain areas of science. At many large universities, over 90% of students enrolled in this course come from these client disciplines. Indeed, all of calculus (and many topics from earlier mathematics courses) come together in this course, and this helps demonstrate how important many of these earlier topics are for students to learn. Of course, there are variants of the ODE course that are aimed at a more mathematically oriented group of students; we include such a syllabus below.

The Computational Approach. This new approach to teaching and learning ODEs has a number of advantages associated with it. A major advantage of the computational approach is that modeling/research projects involving real-life nonlinear systems of ODEs that arise in various areas of science and engineering can now be included as part of the course. As mentioned above, the ODE course often serves as the capstone mathematics course for students in areas of science and engineering, although, of course, most mathematics students go well beyond this course. Given that this is the final mathematics course for many students, it is natural that such students complete their mathematical training by becoming involved in modeling/research topics that are related to their specific major. Almost all such projects involve some sort of numerical experimentation with ODEs that were previously inaccessible to students. Modeling/research projects also develop the important skill of visualizing solutions for ODE students.

These projects can be a team-effort or an individual assignment, and they usually require an extensive written report. Time permitting, oral reports on the project can also be required. Fortunately, almost all ODE textbooks nowadays contain topics for such projects, and there is also plenty of information on modeling applications and lab projects on-line. For example, see the CODEE website.

New Topics to Include. Another advantage of the computational approach is that this allows for the inclusion of topics that have not usually been treated in an introductory ODE course. For example, since visualization of solutions is much more important these days, it is natural to convert second order linear differential equations to planar systems of differential equations, so solutions can then be visualized in the phase plane. Consequently, linear systems of the form $Y' = AY$ are now included in most courses. Usually, only planar systems are covered, so A is a 2 by 2 matrix, although more advanced courses often introduce higher dimensional systems as well. Therefore the new course includes a relatively brief introduction to linear algebraic topics such as matrices, eigenvalues, and eigenvectors. Often these topics are introduced gradually during the course, not necessarily as a week or two week long diversion into linear algebra. The inclusion of these topics connects well with the linear algebra course that many students often take after the ODE course, as it provides a nice visual application of linear algebra. For example, the most important solutions of linear systems are the "straight line" solutions. These are solutions of the form $e^{\lambda t}V$, where λ is an eigenvalue of A and V is its associated eigenvector, so students see early on why eigenvalues and eigenvectors areimportant.

In addition, nonlinear systems are often covered and this necessitates using a qualitative approach to understanding the system as most nonlinear systems cannot be solved analytically. Topics such as nullclines and linearization near equilibria provide some of the tools necessary for this geometric approach. Hamiltonian and gradient systems may also be introduced, and this provides further tools. Some courses even occasionally include modern topics such as chaos and bifurcation theory.

Computational Tools. There is no set standard for how to use computers in the ODE courses. Many institutions use computer algebra systems such as Maple or Mathematica. This is fine when students already have some familiarity with these tools, but often certain students in the course do not come with this background and hence they encounter a steeper learning curve. Another resource is that many of the newer ODE texts come with software specifically designed for topics included in the book. The advantage here is that students do not have to take time to learn how to use the software. A third possibility is the use of spreadsheets. When students try to learn the numerical algorithms for approximating solutions to ODEs (like Euler's Method or Runge-Kutta 4), a natural method to encode these algorithms involves a spreadsheet. Moreover, the graphical capabilities of spreadsheets also help students visualize the outputs of these algorithms. A major advantage of this approach is that almost all students in the ODE course already have a good background using spreadsheets.

What to Eliminate. Given the inclusion mentioned above of more modeling/research projects, topics from linear algebra, and more computational based topics in the contemporary ODE courses, clearly some topics from the traditional ODE course must be dropped. Most modern courses now eliminate some or many of the specialized analytic methods for solving ODEs. For example, integrating factors, variation of parameters, solutions of special equations like the Bernoulli equation, and other such techniques are sometimes not included. Instead, in some instances, the only types of first order ODEs that are now solved analytically are linear and separable equations. Also, series solutions are sometimes eliminated (though, of course, the linearization techniques mentioned above do involve the "first" terms in such a series solution). And, except in courses heavily populated by engineering students, Laplace transforms can also be eliminated.

Cognitive Learning Goals

A. The Culmination of Calculus. ODEs really form the primary basis for the study of calculus and so this course should strive to bring together many of the previously covered concepts in a way that confirms their usefulness and necessity.

B. Use of Technology. The ODE course is easily the course in the introductory undergraduate mathematics curriculum in which the use of technology is most essential. Students should be encouraged to use these tools in homework, in projects, and in simply visualizing the various qualitative aspects of ODEs.

C. Applications. There are major applications involving differential equations in all areas of science and engineering, and so many of these should be included in the ODE course to show students the relevance and importance of this topic. Some applications include mass-spring systems, forced, damped, and undamped pendulum equations, and Newton's Laws (physics), electrical circuits (engineering), enzymatic reactions (chemistry), population models (biology), Kepler's Laws (astronomy), compound interest models (economics), and the Lorenz system (meteorology).

D. Introduction to Higher Level Mathematics. This course also provides an opportunity for students to get a glimpse of some topics in higher level mathematics courses. Examples include linear algebra (solving linear systems of ODEs and linearization), numerical analysis (understanding numerical methods such as Euler's method or RK4 for approximating solutions of ODEs), real analysis (the existence and uniqueness theorem), and dynamical systems (bifurcation theory and chaos).

Sample Syllabi. Because of the variety of aforementioned topics and styles, there is now no one set curriculum for an ODE course. Each of the schools we surveyed had different approaches to the course. Below is a summary of the syllabi of the three types of ODE courses offered at the collegiate level: the captsone college course, the sophomore level service course, and the course aimed primarily at math majors.

1. The capstone ODE course not necessarily aimed at partner disciplines. Many, if not all, community colleges offer Differential Equations as the capstone course in the mathematics department. Some time is devoted to reviewing relevant topics in single variable calculus. The course would not qualify as a replacement for a junior-level introductory ODE course offered at a four-year college or a university. It might, however, qualify as a replacement for a service ODE course offered

in the freshman or sophomore years at those institutions. Here is an outline of a syllabus for such a 13-week semester ODE course.

- First order ODEs: linear equations, radioactive decay model, separable equations, review of relevant topics from single-variable calculus, slope fields, classifying equilibrium points of autonomous first order ODEs, existence and uniqueness theorems, introduction to the use of solvers, Euler's Method, population models. (4 weeks)
- Second order ODEs: solution techniques for linear ODEs with constant coefficients, review of complex numbers, direction fields, mass-spring model, forced and damped harmonic motion, beats and resonance. (4 weeks)
- Planar systems: solution techniques for planar linear systems with constant coefficients, an introduction to relevant topics from linear algebra (matrices, eigenvalues, eigenvectors), nonlinear autonomous systems, classifying equilibrium points, phase portraits, various models. (5 weeks)

2. An ODE course which is part of the math core required of science and engineering students taken in the freshman and sophomore years. That is, this course is really a service course. Unlike the previous syllabus, this course will include many more applications in the client disciplines. As in the earlier course, students will have not taken a course on Linear Algebra prior to this course, so the syllabus allots time to cover the linear algebra topics needed for solving linear systems. Laplace transforms are included. The modeling projects emphasize the usability of ODEs in other disciplines such as biology (predator-prey and competing species models), electrical engineering (circuit theory models), and physics (mechanical systems, the n-body problem). The syllabus for such a 13-week semester course might look something like this:

   - First-order ODEs: Slope fields, separable equations, linear equations, nonlinear ODEs, existence and uniqueness, numerical solutions, qualitative analysis of solutions. (3 weeks)

- Second-order ODEs: Linear ODEs with constant coefficients, forced and damped harmonic oscillator, beats and resonance, phase plane, applications to springs, electrical circuits. (3 weeks)
- Laplace Transforms: Definition, solving initial value problems for linear constant-coefficient second-order ODEs. (2 weeks)
- Linear Systems of ODEs: Solution of constant-coefficient undriven linear systems using techniques of linear algebra, phase plane analysis for linear systems, solution of driven linear constant-coefficient systems. (3 weeks)
- Nonlinear Systems of ODEs: Autonomous systems, planar systems, direction fields, stability of linear and non-linear autonomous systems. (2 weeks)

3. The ODE course as a stand-alone course in the mathematics department taken in the junior year (or perhaps as an honors-level course in place of the aforementioned courses). The syllabus includes all topics in (2.) with the exception of Laplace transforms, and more theory is included throughout the course. Additional topics may include$\otimes$: *thePoincaré−BendixsonTheorem*, *nonlinearplanarautonomoussystems*, *limitcycles*, *stabil*

Possible Textbooks:

Remark: *The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of courses. Please note that some of the books listed below were written by the authors of this report.*

1. Abell, M.L., Braselton, J.P., Introductory Differential Equations, Elsevier, Inc.

2. Blanchard, P., Devaney, R. L., Hall, G. R. Differential Equations. Fourth Edition. Cengage Learning.

3. Borelli, R.L., Coleman, C.S., Differential Equations: A Modeling Perspective, John Wiley and Sons, Inc.

4. Boyce, W.E., DiPrima, R.C., Elementary Differential Equations, John Wiley and Sons, Inc.

5. Braun, M. Differential Equations and Their Applications. Springer.

6. Edwards, H.C., Penney, D.E., Differential Equations: Computing and Modeling, Pearson Publishing

7. Edwards, H.C., Penney, D.E., Elementary Differential Equations with Boundary Value Problems, Pearson Publishing

8. Nagle, R.K., Saff, E.B., Snider, A.D., Fundamentals of Differential Equations, Pearson Publishing

9. Zill, D.G., A First Course in Differential Equations with Modeling Applications, Brooks/Cole, Cengage Learning

More Advanced Textbooks:

10. Hirsch, M. W., Smale, S., Devaney, R. L. Differential Equations, Dynamical Systems, and an Introduction to Chaos. Third Edition. Academic Press.

11. Strogatz, S. Nonlinear Dynamics and Chaos. Westview Press.

Online Resources:

12. The CODEE Project

13. The Boston University ODE Project

14. ODE Software for MATLAB

15. SIMIODE - Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

Articles from Mathematics Education Research:

16. Rasmussen, Chris and Oh Nam Kwon, An inquiry-oriented approach to undergraduate mathematics, *Journal of Mathematical Behavior*26 (2007), 189-194.

    This article appeared in a special issue of the *Journal of Mathematical Behavior*. In addition to providing an overview of the five articles in the issue, the authors highlight the theoretical background for an innovative approach in differential equations called the Inquiry Oriented Differential Equations (IO-DE) project. and provide a summary of two quantitative studies done to assess the effectiveness of the IO-DE project on student learning.