# PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

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*Proposed problems, solutions, and classics should be submitted online at* americanmathematicalmonthly.submittable.com/submit. *Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by September 30, 2024. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

## PROBLEMS

**12461**. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Find all triples $(x, y, z)$ of positive integers such that $x^2 + y^2 = (y\,z - 1)^3$.

**12462**. *Proposed by Tho Nguyen Xuan, Hanoi University of Science and Technology, Hanoi, Vietnam.* What is the minimum value of

$$\left|a+b+c\right|\left(\frac{1}{|a-b|}+\frac{1}{|b-c|}+\frac{1}{|c-a|}\right)$$

over all triples $a, b, c$ of distinct real numbers satisfying $a^2 + b^2 + c^2 = 2(ab + bc + ca)$?

**12463**. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* For a positive integer $n$, let $F_n$ be the $n$th Fibonacci number ($F_0 = 0, F_1 = 1,$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$). Show that when $p$ is prime,

$$\left(\sum_{k=0}^{p-1}\binom{2k}{k}F_k 2^{p-k-1}\right)\left(\sum_{k=0}^{p-1}\binom{2k}{k}^2 8^{p-k-1}\right)$$

is divisible by $p$.

**12464**. *Proposed by Farhood Pouryosefi Kermani, Tehran, Iran.* For a binary string $X$ of length $n$ and an integer $k$ with $1 \le k \le n$, let $\pi_k(X)$ denote the result of reversing the first $k$ elements of $X$ and also the last $n-k$ elements of $X$. For example, $\pi_2(1,0,0,1,1) = (0,1,1,1,0)$. Let $d_k(X)$ be the number of entries in which $X$ and $\pi_k(X)$ differ, and define $A(n)$ to be the maximum of $\min_k d_k(X)/n$ over all choices of $X$.
**(a)** Prove $A(n) \le 1/2$.
**(b)** Prove that $\lim_{n\to\infty} A(n)$ exists.
**(c)** Find the value of $\lim_{n\to\infty} A(n)$.

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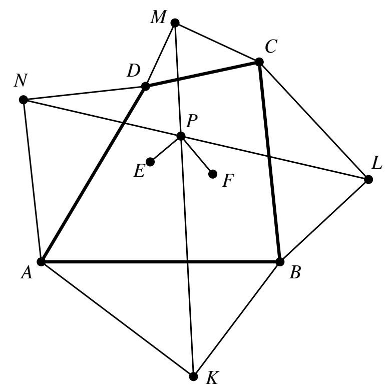
**12465**. *Proposed by Lawrence Glasser, Clarkson University, Potsdam, NY.* Prove

$$\sum_{n=0}^{\infty} \frac{1}{\cosh((4n+2)\pi)} = \frac{\left(\left(2^{1/4}-1\right)^2 \Gamma(1/4)\right)^2}{2^{9/2}\,\pi^{3/2}}.$$

**12466**. *Proposed by Khakimboy Egamberganov, University of Edinburgh, Edinburgh, UK.* Let $ABCD$ be a convex quadrilateral and let $AKB$, $BLC$, $CMD$, $DNA$ be similar right-angled triangles constructed externally to $ABCD$, where $\angle AKB = \angle BLC = \angle CMD = \angle DNA = 90°$ and $\angle KAB = \angle LCB = \angle MCD = \angle NAD$. Let $E$ and $F$ bisect the diagonals $AC$ and $BD$, respectively, and let $P$ be the intersection of $KM$ and $LN$. Prove that $\angle EPF$ is a right angle.




**12467**. *Proposed by Lajos László, Eötvös Loránd University, Budapest, Hungary.* Given any real number $c$, it is not hard to see that there is a unique differentiable function $s : [1,\infty) \to \mathbb{R}$ such that (1) $s(n) = 1/n$ for all positive integers $n$, (2) $s$ is quadratic or linear on $[n, n+1]$ for all positive integers $n$, and (3) the right derivative of $s$ at 1 is $c$. (A function satisfying (1) and (2) is a *quadratic spline*.) For what values of $c$ is $s$ decreasing and convex?

# SOLUTIONS

### A Nilpotent Commutator

**12339** [2022, 686]. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let $A$ and $B$ be complex $n$-by-$n$ matrices for which $A^2 + xB^2 = y(AB - BA)$, where $x$ is a positive real number and $y$ is a real number such that $(1/\pi)\cos^{-1}\left((y^2-x)/(y^2+x)\right)$ is irrational. Prove that $(AB-BA)^n$ is the zero matrix.

*Solution by Kyle Gatesman, Fairfax, VA.* Let $U = A + i\sqrt{x}B$ and $V = A - i\sqrt{x}B$. Note that $y \pm i\sqrt{x} \neq 0$ because $y$ is real and $x$ is positive. Since

$$UV = A^2 + xB^2 - i\sqrt{x}(AB - BA) = (y - i\sqrt{x})(AB - BA)$$

and

$$VU = A^2 + xB^2 + i\sqrt{x}(AB - BA) = (y + i\sqrt{x})(AB - BA),$$

we have

$$VU = \frac{y + i\sqrt{x}}{y - i\sqrt{x}}\,UV = \frac{y^2 - x + 2yi\sqrt{x}}{y^2 + x}\,UV.$$

Let $(y + i\sqrt{x})/(y - i\sqrt{x}) = \cos\theta + i\sin\theta = e^{i\theta}$. The spectrum of $VU$ is $e^{i\theta}$ times that of $UV$. By hypothesis, $\theta$ is not a rational multiple of $\pi$, so $e^{in\theta} \neq 1$ for all nonzero integers $n$.

It is well known for complex $n$-by-$n$ matrices $U$ and $V$, that $UV$ and $VU$ have the same characteristic polynomial. Hence any eigenvalue of $UV$ or $VU$ is an eigenvalue of the other. Thus the spectrum of $UV$ is invariant under multiplication by $e^{i\theta}$. Since the complex numbers $e^{i\theta}, e^{2i\theta}, e^{3i\theta}, \dots$ are distinct and the spectrum of $UV$ has cardinality at most $n$, we conclude that the only eigenvalue of $UV$ is zero. It follows that the characteristic polynomial of $AB - BA$ is $\lambda^n$. By the Cayley–Hamilton Theorem, $(AB - BA)^n$ is the zero matrix.

Also solved by C. P. Anil Kumar (India), S. Bhadra, E. A. Herman, O. P. Lossers (Netherlands), M. Omarjee (France), R. Stong, L. Zhou, and the proposer.

## A Nascent Delta Function

**12340** [2022, 686]. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g\colon [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = Cg(1/2)$$

for some constant $C$ (independent of $g$), and determine the value of $C$.

*Solution by Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.* Substituting $u = n(2x - 1)$ and letting $\chi_{[-n,n]}$ denote the characteristic function of $[-n, n]$ gives

$$\frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{g\left(\frac{1}{2} + \frac{u}{2n}\right) \chi_{[-n,n]}(u)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n}\, du.$$

Since $g$ is continuous, we may choose a $K > 0$ such that $|g(x)| \le K$ on $[0, 1]$. Further, for $n \ge 2$, the binomial theorem gives

$$\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n \ge 2\left(1 + \binom{n}{2}\frac{u^2}{n^2}\right) \ge 2\left(1 + \frac{u^2}{4}\right).$$

Therefore for $n \ge 2$,

$$\frac{1}{2}\left|\frac{g\left(\frac{1}{2} + \frac{u}{2n}\right) \chi_{[-n,n]}(u)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n}\right| \le \frac{K}{4 + u^2}.$$

This upper bound has finite integral, so the dominated convergence theorem applies, and we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{g\left(\frac{1}{2} + \frac{u}{2n}\right)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n}\, du \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{g(1/2)}{e^u + e^{-u}}\, du \\
&= \frac{1}{2} g(1/2) \arctan(e^u)\Big|_{-\infty}^{\infty} = \frac{\pi}{4} g(1/2).
\end{aligned}$$

Also solved by M. Aassuka (France), A. Berkane (Algeria), S. Bhadra (India), H. Chen (US), W. J. Cowieson, M.-C. Fan (China), K. Gatesman, R. Guadalupe (Philippines), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), E. J. Ionascu, S. Kaczkowski, O. Kouba (Syria), C. Krattenthaler (Germany), G. Lavau (France), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), F. Masroor, R. Mortini (Luxembourg) & R. Rupp (Germany), M. Omarjee (France), D. Pascuas (Spain), P. Perfetti (Italy), K. Schilling, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, J. Vukmirović (Serbia), J. H. Yan (China), and the proposer.

### A Product Inequality

**12341** [2022, 686]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let $x_1, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i^2 \le n$, and let $S = \sum_{i=1}^n x_i$. Prove

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i x_{i+1}}\right)^{x_i^2} \ge 2^{S^2/n},$$

where $x_{n+1}$ is taken to be $x_1$.

*Solution by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* We prove the more general inequality

$$\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2} \ge \left(1 + \frac{n}{T}\right)^{S^2/n}, \qquad (*)$$

where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are positive real numbers, $S = \sum_{i=1}^n x_i$, and $T = \sum_{i=1}^n y_i$. The required inequality follows from $(*)$ by letting $y_i = x_i x_{i+1}$ and noting that, by the rearrangement inequality,

$$T = \sum_{i=1}^n y_i = \sum_{i=1}^n x_i x_{i+1} \le \sum_{i=1}^n x_i^2 \le n.$$

To prove $(*)$, we compute

$$\log\left(\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2}\right) = \sum_{i=1}^n x_i^2 \log\left(1 + \frac{1}{y_i}\right)$$
$$= \sum_{i=1}^n x_i^2 \int_0^1 \frac{dt}{y_i + t} = \int_0^1 \sum_{i=1}^n \frac{x_i^2}{y_i + t}\, dt.$$

For $0 \le t \le 1$, the Cauchy–Schwarz inequality implies

$$S^2 = \left(\sum_{i=1}^n \sqrt{y_i + t} \cdot \frac{x_i}{\sqrt{y_i + t}}\right)^2 \le \sum_{i=1}^n (y_i + t) \cdot \sum_{i=1}^n \frac{x_i^2}{y_i + t} = (T + nt) \sum_{i=1}^n \frac{x_i^2}{y_i + t},$$

so

$$\sum_{i=1}^n \frac{x_i^2}{y_i + t} \ge \frac{S^2}{T + nt}.$$

Therefore

$$\log\left(\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2}\right) = \int_0^1 \sum_{i=1}^n \frac{x_i^2}{y_i + t}\, dt \ge \int_0^1 \frac{S^2}{T + nt}\, dt = \frac{S^2}{n} \log\left(1 + \frac{n}{T}\right).$$

Inequality $(*)$ follows.

Also solved by P. Bracken, W. J. Cowieson, O. P. Lossers (Netherlands), S. Patra, A. Stadler (Switzerland), R. Stong, and the proposer.

### Characterizing Cyclic Quadrilaterals

**12343** [2022, 785]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Prove that $ABCD$ is a cyclic quadrilateral (i.e., the four vertices lie on a circle) if and only if

$$\frac{f^2 - e^2}{ac + bd} = \frac{\left(a^2 - c^2\right)\left(b^2 - d^2\right)}{(ab + cd)(ad + bc)}.$$

*Solution by Prithwijit De, Mumbai, India.* Denote the angles of $ABCD$ at the four vertices by $A$, $B$, $C$, and $D$. Let

$$T_1 = \cos A + \cos C = \frac{d^2 + a^2 - f^2}{2ad} + \frac{b^2 + c^2 - f^2}{2bc},$$
$$T_2 = \cos B + \cos D = \frac{a^2 + b^2 - e^2}{2ab} + \frac{c^2 + d^2 - e^2}{2cd}.$$

Algebraic manipulation yields

$$2abcd\big((ab + cd)T_1 - (ad + bc)T_2\big) = (ac + bd)(a^2 - c^2)(b^2 - d^2) - (ab + cd)(ad + bc)(f^2 - e^2).$$

It therefore suffices to show that $ABCD$ is cyclic if and only if

$$(ab + cd)T_1 - (ad + bc)T_2 = 0.$$

By the sum-to-product formula for the cosine function and the fact that $B + D = 2\pi - (A + C)$, we have

$$(ab + cd)T_1 - (ad + bc)T_2 = 2\left((ab + cd)\cos\left(\frac{A - C}{2}\right) + (ad + bc)\cos\left(\frac{B - D}{2}\right)\right)\cos\left(\frac{A + C}{2}\right).$$

Since $|A - C|$ and $|B - D|$ are less than $\pi$, the values $\cos((A - C)/2)$ and $\cos((B - D)/2)$ are strictly positive. Hence $(ab + cd)T_1 - (ad + bc)T_2 = 0$ if and only if $\cos((A + C)/2) = 0$, which happens if and only if $A + C = \pi$, which is equivalent to $ABCD$ being cyclic.

Also solved by G. Fera (Italy), O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. Hodges (UK), O. P. Lossers (Netherlands), C. R. Pranesachar (India), C. Schacht, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer.

## Linear Combinations of Powers That Are Not Perfect Squares

**12346** [2022, 785]. *Proposed by Nguyen Quang Minh, Hwa Chong Institution, Bukit Timah, Singapore.* Prove that there are infinitely many integers $A$ such that, for every nonzero integer $x$ and distinct positive odd integers $m$ and $n$, the integer $x^m + Ax^n$ is not a perfect square.

*Solution by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* We claim that the infinite family consisting of the negatives of primes congruent to 3 modulo 8 satisfies the requirements of the problem.

Let $A = -p$ for such a prime $p$. Factoring out the perfect square $x^{\min\{m,n\}-1}$, we see that it suffices to show that no $x^m - px^n$ is a perfect square when $m$ and $n$ are odd and either $m = 1$ or $n = 1$. Suppose otherwise.

First consider $m = 1$ and set $k = (n - 1)/2$. With $x - px^n = x(1 - px^{2k})$, both factors are negative. Since also $1 - px^{2k}$ is relatively prime to $x$, both $-x$ and $px^{2k} - 1$ must be squares. Modulo $p$, the equation $px^{2k} - 1 = a^2$ for a positive integer $a$ reduces to $a^2 \equiv -1$. However, when $p \equiv 3 \pmod 8$ (indeed, whenever $p \equiv 3 \pmod 4$) the value $-1$ is not a square modulo $p$, a contradiction.

Now consider $n = 1$ and set $k = (m - 1)/2$, so $x^m - px = x(x^{2k} - p)$. The greatest common divisor of $x$ and $x^{2k} - p$ is 1 or $p$. Since $x^m - px$ is a square, we have either (i) $x = \pm a^2$ and $x^{2k} - p = \pm b^2$ or (ii) $x = \pm pa^2$ and $x^{2k} - p = \pm pb^2$, for some integers $a$ and $b$.

Note that squares are congruent to 0, 1, or 4 modulo 8, and recall that $p \equiv 3 \bmod 8$. In case (i), if $a$ is odd, then $x^{2k} - p \equiv 6 \pmod 8$. If $a$ is even, then $x^{2k} - p \equiv 5 \pmod 8$. In both subcases, this value cannot be a square or its negative, so we move on to case (ii). Substituting for $x$ and simplifying, we have $p^{2k-1}a^{4k} - 1 = \pm b^2$. The left side is positive. However, again because $-1$ cannot be a square modulo $p$, the alternative $p^{2k-1}a^{4k} - 1 = b^2$ is also impossible.

*Editorial comment.* All solvers had roughly similar approaches. We generalize some of their families. Using the fact that $-2$ is a quadratic nonresidue for primes $p$ congruent to 5 or 7 modulo 8, one can show that the family $A = p^r$ satisfies the condition of the problem for such primes $p$ and even $r$. Another family is given by $A = p^r$, where $p$ is a prime congruent to 7 modulo 16 and $r$ is odd. This can be proved by the method of descent.

Also solved by J. Boswell & C. Curtis, W. J. Cowieson, K. Gatesman, P. W. Lindstrom, R. Stong, R. Tauraso (Italy), H. von Eitzen (Germany), and the proposer.

### A Functional Equation With Piecewise Linear Solutions

**12347** [2022, 786]. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania.* Let $a$ and $b$ be real numbers with $0 < a < 1 < b$. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

*Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* We show that there are exactly four solutions, given by

$$f(x) = ax, \quad f(x) = bx, \quad f(x) = \begin{cases} ax, & \text{if } x \ge 0, \\ bx, & \text{if } x < 0, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} bx, & \text{if } x \ge 0, \\ ax, & \text{if } x < 0. \end{cases}$$

Clearly these four functions are solutions. Now let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$. For all $x \in \mathbb{R}$,

$$x = \frac{(a+b)f(x) - f(f(x))}{ab}.$$

This implies that $x = y$ if $f(x) = f(y)$, so $f$ is one-to-one. Since $f$ is continuous, it follows that $f$ is monotonic, and consequently $f \circ f$ is increasing. Moreover, the equality

$$f(x) = \frac{f(f(x)) + abx}{a+b} \tag{1}$$

shows that $f$ is increasing. Since $f(0) = 0$, the sign of $f(x)$ is the same as the sign of $x$. By (1), we have $f(x) > abx/(a+b)$ for all $x > 0$ and $f(x) < abx/(a+b)$ for all $x < 0$. This implies that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Hence $f$ is bijective.

Let $g = f^{-1}$. Applying the functional equation to $g(g(x))$ leads to

$$g(g(x)) - \left(\frac{1}{a} + \frac{1}{b}\right) g(x) + \frac{1}{ab} x = 0.$$

Thus, $g$ satisfies the same functional equation as $f$, but with $a$ and $b$ replaced by $1/a$ and $1/b$.

Suppose $x > 0$. We define two sequences $\{x_n\}_{n\ge0}$ and $\{y_n\}_{n\ge0}$ by $x_0 = x$, $y_0 = f(x)$, and $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$ when $n \ge 0$. By the functional equations of $f$ and $g$, $\{x_n\}_{n\ge0}$ and $\{y_n\}_{n\ge0}$ satisfy the following second-order linear recurrence relations:

$$x_0 = x, \qquad x_1 = f(x), \qquad x_{n+2} - (a+b)x_{n+1} + abx_n = 0,$$

$$y_0 = f(x), \qquad y_1 = x, \qquad y_{n+2} - \left(\frac{1}{a} + \frac{1}{b}\right) y_{n+1} + \frac{1}{ab} y_n = 0.$$

Solving these recurrence relations, we find that for all $n \geq 0$,

$$x_n = \frac{f(x) - bx}{a - b} a^n + \frac{f(x) - ax}{b - a} b^n, \tag{2}$$

$$y_n = \frac{f(x) - bx}{a - b} a^{1-n} + \frac{f(x) - ax}{b - a} b^{1-n}. \tag{3}$$

We now consider two cases. If $f(x) \leq x$, then because $f$ is increasing, we have $x_n \geq x_{n+1} > 0$ for all $n$. Thus the sequence $(x_n)_{n\geq 0}$ is nonincreasing and bounded below, so it must be convergent. Since $b > 1$, the coefficient of $b^n$ in (2) must be zero, which implies that $f(x) = ax$.

On the other hand, if $f(x) > x$, then similar reasoning shows that the sequence $(y_n)_{n\geq 0}$ converges, the coefficient of $a^{1-n}$ in (3) is zero, and $f(x) = bx$.

Thus, for all $x > 0$, either $f(x) = ax$ or $f(x) = bx$, so $f(x)/x$ can take only the two values $a$ and $b$ on $(0, \infty)$. However, since $f$ is continuous, it cannot take both values. We conclude that either $f(x) = ax$ for all $x > 0$ or $f(x) = bx$ for all $x > 0$.

Applying the above analysis for $x > 0$ to the function $-f(-x)$, we conclude that either $f(x) = ax$ for all $x < 0$ or $f(x) = bx$ for all $x < 0$. Thus there are no solutions other than the four listed earlier.

Also solved by J. Boswell & C. Curtis, H. Chen (China), W. J. Cowieson, H. von Eitzen (Germany), D. Henderson, N. Hodges (UK), O. P. Lossers (Netherlands), R. Mortini (Luxembourg), K. Schilling, R. Stong, R. Tauraso (Italy), and the proposer.

## A Variation on the Josephus Problem

**12348** [2022, 786]. *Proposed by Erik Vigren, Uppsala, Sweden, and Hans Rullgård, Kungälv, Sweden.* We have $n$ people in a circle, numbered from 1 to $n$ clockwise. They are removed one at a time as follows, until just one remains. At each step, remove the $n$th person among those remaining, where the count starts at the lowest-numbered person remaining and proceeds clockwise. Let $W(n)$ be the number of the last person remaining. For example, with $n = 5$, we remove in order the people numbered 5, 1, 3, and 2, and so $W(5) = 4$. (This is a variation of the classic Josephus problem.)
**(a)** What is $W(10^{12})$?
**(b)** For $n \geq 5$, show that $W(n) = n - 4$ if and only if $n/2$ is a Sophie Germain prime (i.e., $n/2$ and $n + 1$ are prime).
**(c)** Find the smallest even number that does not equal $W(n)$ for any $n$.

*Composite solution by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy, and the proposers.*
**(a)** By reversing the procedure, we show $W\left(10^{12}\right) = 671{,}046{,}354{,}072$. As in the problem statement, the *number* of a person is that person's original index and remains unchanged. The *position* of a person at a given time is that person's index among the remaining people; it counts the remaining people with smaller numbers (plus 1).

Consider the point in the process when $m$ people remain. In the next step, skipping $n - 1$ people means passing through the entire list $r$ times before stopping at the person to be removed, where $r = \lfloor (n-1)/m \rfloor$. The person removed will be in position $n - rm$. We say that removals whose associated value of $r$ are the same occur in the same *round*, and we label this round with the value $r$. For example, in round 0 we remove person $n$, and in round 1 we remove all the remaining odd-numbered people, starting with person 1. The rounds occur in increasing order, but the round numbers are not consecutive. For example, when $n = 9$ there is no round 3, because $\lfloor 8/3 \rfloor = 2$ and $\lfloor 8/2 \rfloor = 4$. Rather than reversing the procedure one removal at a time, the computation is quicker if we reverse it one round at a time. This will also be useful in part **(c)**.

Now consider the time when a round has just been finished and $k$ rounds remain to be completed. Let $m_k$ denote the number of people remaining at this time, and let $p_k$ denote the position at this time of the person $P$ who will be the last person remaining. Thus $m_0 = 1$ and $p_0 = 1$, since $P$ is never removed. For $k \geq 1$, let $r_k$ denote the number of the round about to start. By definition, $r_k = \lfloor (n-1)/m_k \rfloor$.

The last removal in round $r_{k+1}$ occurs with $m_k + 1$ people remaining, so

$$r_{k+1} = \lfloor (n-1)/(m_k+1) \rfloor. \tag{1}$$

When $r_{k+1} > 0$, the number of people remaining at the start of round $r_{k+1}$ is the largest $m$ such that $r_{k+1} = \lfloor (n-1)/m \rfloor$; that is,

$$m_{k+1} = \lfloor (n-1)/r_{k+1} \rfloor. \tag{2}$$

During round $r_{k+1}$, when $m$ people remain, the person in position $n - r_{k+1}m$ will be removed. This position strictly increases throughout round $r_{k+1}$ as $m$ decreases from $m_{k+1}$ to $m_k + 1$. Meanwhile, the position of $P$ decreases from $p_{k+1}$ to $p_k$. Since $P$ reaches $p_k$, the position of $P$ must decrease on the step that starts with $m$ people remaining if and only if

$$n - r_{k+1}m \leq p_k. \tag{3}$$

By (2), we have $(n-1)/r_{k+1} < m_{k+1} + 1$, which yields $n - r_{k+1}(m_{k+1}+1) < 1$. Also, the definition of $r_k$ implies $(n-1)/m_k \geq r_k \geq r_{k+1} + 1$, from which we obtain $n - r_{k+1}m_k \geq m_k + 1$. Together, these inequalities yield

$$n - r_{k+1}(m_{k+1}+1) < 1 \leq p_k < m_k + 1 \leq n - r_{k+1}m_k.$$

It follows that there is some integer $j$ with $0 \leq j \leq m_{k+1} - m_k$ such that

$$n - r_{k+1}(m_{k+1} - (j-1)) \leq p_k < n - r_{k+1}(m_{k+1} - j).$$

By (3), there will then be exactly $j$ steps during round $r_{k+1}$ on which the position of $P$ decreases by $1$. Therefore,

$$p_{k+1} = p_k + j = p_k + \left\lfloor \frac{p_k + r_{k+1}\,(m_{k+1}+1) - n}{r_{k+1}} \right\rfloor. \tag{4}$$

We now have a recursive procedure, starting from $m_0 = p_0 = 1$. Given $m_k$ and $p_k$, we use $m_k$ to compute $r_{k+1}$ by (1), $r_{k+1}$ to compute $m_{k+1}$ by (2), and then all of $\{p_k, r_{k+1}, m_{k+1}\}$ to compute $p_{k+1}$ by (4). We run the recursion until reaching $k$ such that $m_k$ equals $n-1$. The original position (and number) of $P$ is then $p_k$. In the particular instance $n = 10^{12}$, we obtain $k = 1999997$, leading to $W(n)$ as claimed.

**(b)** Assume $n \geq 5$. Because all people with odd numbers will have been removed by the end of round 1, $W(n)$ is an even number less than $n$. In particular, $n-4$ is removed by then if $n$ is odd, so we need only consider even $n$. When $n$ is even, the person with the larger number will be removed when only two people remain. Therefore, $W(n) = n-4$ if and only if the last two people are numbered $n-4$ and $n-2$.

Suppose that $m$ people remain, where $m \leq n/2 - 1$. Recall that $n$ is removed first and then all odd numbers. If both $n-4$ and $n-2$ remain, then they occupy positions $m-1$ and $m$. To avoid removing either, $n$ must not be congruent to $m-1$ or $m$ modulo $m$. That is, we avoid removing person $n-2$ if and only if $n$ is not divisible by any number from 3 to $n/2-1$, meaning that $n/2$ is prime. Similarly, we avoid removing person $n-4$ if and only if $n-1$ is not divisible by any number from 3 to $n/2-1$, meaning that $n+1$ is prime.

**(c)** We show that the smallest even number that does not equal $W(n)$ for any $n$ is 34. The table below gives the smallest value of $n$ yielding each value of $W(n)$ less than 34, by explicit computation.

| $W(n)$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 3 | 5 | 7 | 16 | 11 | 13 | 50 | 17 | 19 | 76 | 23 | 56 | 248 | 29 | 31 | 424 |

We need only consider $n > 34$ and show that in all cases person 34 is removed at some point in the process. We have observed that person $n$ is removed in round 0, and all smaller odd numbers are removed in round 1. Person 34 is then in position 17.

Since round $r$ is defined as $\{m: \lfloor (n-1)/m \rfloor = r\}$, the number of people remaining when round $r$ ends is $\min\{m: \lfloor (n-1)/m \rfloor = r\} - 1$. This number is $\lfloor (n-1)/(r+1) \rfloor$. Let $a_{r+1}$ be the integer such that

$$\lfloor (n-1)/(r+1) \rfloor = (n - a_{r+1})/(r+1).$$

The first person removed in round $r+1$ is in position $a_{r+1}$ at the start of the round. For each subsequent removal in round $r+1$, the removed element pushes the round-starting position of the next person removed up by $r+2$. That is, the key additional observation is that positions at the start of round $r+1$ of the people removed in round $r+1$ are

$$a_{r+1}, \quad a_{r+1} + r + 2, \quad a_{r+1} + 2r + 4, \dots .$$

For even $n$, those removed in round 2 start the round in positions 2, 5, 8, 11, 14, 17, .... Hence we may assume $n$ is odd.

For odd $n$, those removed in round 2 start the round in positions 1, 4, 7, 10, 13, 16, .... Thus after round 2, person 34 is in position 11.

When $n \equiv 3 \pmod 6$, those removed in round 3 start the round in positions 3, 7, 11, 15, ..., so we may forbid this case.

When $n \in \{1, 5, 7, 11\} \pmod{12}$, getting $(n - a_3)/3$ to be an integer requires $a_3 \in \{1, 2\}$. Those removed in round 3 start the round in positions 1, 5, 9, 13, ..., or positions 2, 6, 10, 14, .... In both cases, person 34 ends round 3 in position 8.

When $n \in \{7, 11\} \pmod{12}$, we have $a_4 = 3$, and those starting round 4 in positions 3, 8, ... are removed. Hence we may forbid this case.

When $n \in \{1, 5\} \pmod{12}$, we have $a_4 = 1$, and those starting round 4 in positions 1, 6, ... are removed. Hence person 34 occupies position 6 at the end of round 4. Since $a_5 \in \{1, 2, 3, 4, 5\}$, round 5 removes exactly one person from the first five positions, so person 34 ends round 5 in position 5.

When $n \equiv 5 \pmod{12}$, we have $a_6 = 5$, so round 6 removes person 34.

Hence we may assume $n \equiv 1 \pmod{12}$. If also $n \ge 73$, then at least 12 people remain at the end of round 5. When the number of people remaining is in $\{12, 6, 4, 3, 2\}$, the person occupying the first position at that time will be removed. This means that person 34, who is already as early as position 5 when at least 12 people remain, is removed while a person still remains.

To complete the proof, it remains only to check explicitly that $W(n) \neq 34$ when $n \in \{37, 49, 61\}$.

*Editorial comment.* Reasoning like that for part **(b)** shows that $W(n) = n - 1$ if and only if $n$ is an odd prime. Round $r$ actually eliminates one or more people if $(n-1)/(r+1) < \lfloor (n-1)/r \rfloor$. This holds for all $r$ with $r \le r^*$, where $r^* = \lfloor (\sqrt{4n-3} - 1)/2 \rfloor$. Thereafter, at most one person is removed per round. As a result, the number of rounds in which people are removed is $r^* + \lfloor (n-1)/(r^*+1) \rfloor$.

Also solved by O. P. Lossers (Netherlands). Parts **(b)** and **(c)** also solved by K. Schilling and Eagle Problem Solvers.

### A Lobachevsky-type Formula

**12351** [2022, 886]. *Proposed by Seán Stewart, King Abdullah University of Science and*

*Technology, Thuwal, Saudi Arabia.* Evaluate

$$\int_0^\infty \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx.$$

*Solution by Mohammed Aassila, Strasbourg, France.* Let $I$ denote the requested integral. We prove that

$$I = -\frac{\pi}{4}\left(\ln 2 + \frac{\ln(1+\sqrt{3})}{\sqrt{3}}\right).$$

We have

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^\infty \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx = \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx \\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_0^\pi \frac{(-1)^k\ln\left(\cos^2 x\right)\sin^3 x}{(x+k\pi)^3\left(1+2\cos^2 x\right)}\,dx \\ &= \frac{1}{2}\int_0^\pi \left(\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(x+k\pi)^3}\right)\frac{\ln\left(\cos^2 x\right)\sin^3 x}{1+2\cos^2 x}\,dx, \end{aligned}$$

where the final interchange of integration and summation can be justified by the dominated convergence theorem.

To evaluate the summation in the last formula, we start with the equation

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{x+k\pi} = \frac{1}{\sin x}.$$

(See I. S. Gradshteyn, I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, equation 1.422.6.) Differentiating twice, we get

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(x+k\pi)^3} = \frac{1+\cos^2 x}{2\sin^3 x},$$

so this gives

$$\begin{aligned} I &= \frac{1}{4}\int_0^\pi \frac{(1+\cos^2 x)\ln\left(\cos^2 x\right)}{1+2\cos^2 x}\,dx = \int_0^{\pi/2}\frac{(1+\cos^2 x)\ln\left(\cos x\right)}{1+2\cos^2 x}\,dx \\ &= \frac{1}{2}\int_0^{\pi/2}\ln(\cos x)\,dx + \frac{1}{2}\int_0^{\pi/2}\frac{\ln\left(\cos x\right)}{1+2\cos^2 x}\,dx. \end{aligned}$$

Both of these integrals are special cases of equation 4.385.3 in Gradshteyn and Ryzhik:

$$\int_0^{\pi/2}\frac{\ln(\cos x)}{b^2\sin^2 x + a^2\cos^2 x}\,dx = \frac{\pi}{2ab}\ln\left(\frac{b}{a+b}\right)$$

for $a, b > 0$. Applying this with $b = 1$ and both $a = 1$ and $a = \sqrt{3}$ leads to the claimed answer.

*Editorial comment.* As several solvers noted, the beginning of this argument proves a Lobachevsky-type result: For any continuous function $f(x)$ that is periodic with period $\pi$,

$$\int_{-\infty}^\infty \frac{\sin^3 x}{x^3}f(x)\,dx = \frac{1}{2}\int_0^\pi (1+\cos^2 x)f(x)\,dx.$$

Also solved by T. Amdeberhan, A. Berkane (Algeria), P. Bracken, B. Bradie, C. Burnette, H. Chen (US), B. E. Davis, M. L. Glasser, G. C. Greubel, N. Hodges (UK), W. Janous (Austria), L. Kempeneers & J. V. Casteren (Belgium), O. Kouba (Syria), K. Nelson, M. Omarjee (France), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), Y. Zhang (China), and the proposer.

# CLASSICS

**C25.** Let $w_0, w_1, \ldots$ be the sequence of *Fibonacci words*, defined by $w_0 = 0$, $w_1 = 1$, and, for $n \geq 2$, $w_n = w_{n-2}w_{n-1}$, the concatenation of $w_{n-2}$ and $w_{n-1}$. Thus the sequence begins $0, 1, 01, 101, 01101, 10101101, 0110110101101, \ldots$. Show that, for $n \geq 3$, removing the first two symbols from $w_n$ yields a palindrome.

### The Tennis Ladder

**C24.** *Due to Colin L. Mallows.* Over the history of a certain tennis club, every player has played at least one match against every other player. Matches are played one at a time, and after each match a ranking of the players in the club is computed as follows. Starting with the most recent match and working backwards through time, use the match results to build up a partial order. Ignore any match that is inconsistent with more recent results. The final result is guaranteed to be a linear order, since any incomparability between a pair of players is resolved when a match between them is encountered. This linear order becomes the new club ranking. Prove or disprove: A player cannot rise in the club ranking by intentionally losing a match.

*Solution.* The assertion is false. Suppose that the results of the last nine matches among six players are as follows, where we write $a > b$ for a match where player $a$ defeats player $b$ and we list the matches from oldest to most recent.

$$2{>}3,\ \ 6{>}1,\ 2{>}4,\ 1{>}2,\ 6{>}4,\ 4{>}5,\ 3{>}4,\ 3{>}6,\ 5{>}6$$

The ranking at this moment is $1 > 2 > 3 > 4 > 5 > 6$, with player 3 in third place. However, if player 3 loses the next match to player 5, the ranking becomes $5 > 3 > 6 > 1 > 2 > 4$, with player 3 in second place. So player 3 ranks higher after losing.

*Editorial comment.* The problem appeared as E3240 [1987, 996; 1989, 530] in this MONTHLY. The problem statement has two interpretations. The strong form asks if a player can rank higher immediately after throwing a match. The weak form asks if a player can rank higher today by deciding to forfeit a match that took place in the past. No solution to the strong form of the problem was received from the MONTHLY readership other than the proposer's solution, which involved seven players. The example here involves six players. This raises the question of whether there is an example with five players.

One can show that any time a player defeats a lower-ranked opponent (or loses to a higher-ranked opponent), the ranking remains unchanged. However, reversing the outcome of each match in the example above shows that defeating a higher-ranked opponent can lower one's overall ranking.

Say that a ranking algorithm *respects duality* if changing all wins to losses reverses the resulting ranking. A familiar algorithm for ranking tennis club members is as follows: If a lower-ranked player A defeats a higher-ranked player B, the new ranking is formed by replacing B with A in the prior ranking and moving B and all the players ranked between A and B down one spot. If a higher-ranked player defeats a lower-ranked player, the ranking remains unchanged. One concern with this usual algorithm is that it fails to respect duality. The algorithm of this problem is an alternative that does respect duality. The existence of the example above, however, shows that this ranking system violates a certain kind of monotonicity and suggests that it is an unreasonable system for actual use.